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A COMPLETE BOUNDARY INTEGRAL FORMULATION FOR STEADY COMPRESSIBLE INVISCID FLOWS GOVERNED BY NON-LINEAR EQUATIONS

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SUMMARY

A complete boundary integral formulation for steady compressible inviscid flows governed by non-linear equations is established by using the specific mass flux as a dependent variable. Thus, the dimensionality of the problem to be solved is reduced by one and the computational mesh to be generated is needed only on the boundary of the domain. It is shown that the boundary integral formulation developed in this paper is equivalent to the results of distributions of the fundamental solutions of the Laplacian operator equation with a different order along the boundaries of the domain. Hence, we have succeeded in establishing the fundamental-solution method for compressible inviscid flows governed by non-linear equations.

KEY WORDS Compressible flows Boundary integral equation method Fundamental-solution method Non-linear potential equation

INTRODUCTION

The boundary integral representations for the solution of linear equations in fluid mechanics have been used widely. The boundary integral formulation reduces the dimensionality of problem to be solved by one and the computational mesh to be generated is needed only on the boundaries of the domain. Thus, the required computer storage and computing time will be reduced greatly. At the present time, the boundary integral formulation for steady compressible inviscid flows governed by non-linear equations has not yet been well developed; none of existing methods, except Reference 1, avoid the appearance of domain integrals in their integral representations. The boundary integral formulation in Reference 1 is based on the analytical continuation into the complex plane and, thus, it is suitable only for two-dimensional flows. In this paper a complete boundary integral formulation for two- and three-dimensional steady compressible non-linear potential equation by using the specific mass flux $\rho\mathbf{V}$ as a dependent variable is established and the dimensionality of the problem to be solved is reduced exactly by one.

THEORETICAL BASIS

The governing equation for steady compressible inviscid flows can be written as

$$\nabla\cdot\rho\mathbf{q}=0 \quad \text{and} \quad \nabla\times\rho\mathbf{q}=\boldsymbol{\omega}_{\mathrm{f}}. \tag{1}$$

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If we consider that the flow field is isentropic, we have

$$\frac{\rho}{\rho_\infty}=\left[1+\frac{(k-1)}{2}Ma_\infty^2(1-q^2)\right]^{1/(k-1)}, \tag{2}$$

$$C_p=\frac{p-p_\infty}{(1/2)\rho_\infty q_\infty^2}=\frac{2}{kMa_\infty^2}\left\{\left[1+\frac{(k-1)}{2}Ma_\infty^2(1-q^2)\right]^{k/(k-1)}-1\right\}, \tag{3}$$

where $\mathbf{q}$, ρ, p, C_p and Ma represent total velocity, density, pressure, pressure coefficient and the Mach number of the flow field, respectively; k is the specific heat ratio of fluid and $\boldsymbol{\omega}_f$ is the curl of $\rho\mathbf{q}$. The subscript ∞ denotes the corresponding free-stream value. Obviously, equation (1) is non-linear for variable $\mathbf{q}$. Let

$$\rho\mathbf{q}=\rho\mathbf{V}+\rho_\infty\mathbf{q}_\infty, \tag{4}$$

where $\rho\mathbf{V}$ is the perturbational specific mass flux vector for $\rho_\infty\mathbf{q}_\infty$.

Substituting equation (4) into equation (1), one obtains the governing equation for $\rho\mathbf{V}$ as follows:

$$\nabla\cdot\rho\mathbf{V}=0, \qquad \nabla\times\rho\mathbf{V}=\boldsymbol{\omega}_f. \tag{5}$$

The boundary conditions are

$$\rho\mathbf{V}=0 \quad \text{at the far field } \rho\mathbf{q}=\rho_\infty\mathbf{q}_\infty, \tag{6a}$$

$$\rho\mathbf{V}\cdot\mathbf{n}=-\rho_\infty\mathbf{q}_\infty\cdot\mathbf{n} \quad \text{on the body surface } \rho\mathbf{q}\cdot\mathbf{n}=0, \tag{6b}$$

where $\mathbf{n}$ is the unit outnormal vector of the boundary. For lifting flow, a trailing-edge condition must also be added. It can be seen from equations (5) and (6) that these equations are formally linear for variable $\rho\mathbf{V}$. Since $\rho\mathbf{V}$ is the perturbational mass flux, no jumping phenomenon for $\rho\mathbf{V}$ will occur even in the flow field embedded with normal shock wave. Hence, there is no special difficulty for transonic flows. Following the general expressions of Green's formula[2] for vector field the integral representation for equation (5) can be written as

$$T\rho\mathbf{V}(\mathbf{r})=\int_\Omega \nabla_0 G\times(\nabla_0\times\rho_0\mathbf{V}_0)\,\mathrm{d}\Omega_0+\int_B(\rho_0\mathbf{V}_0\cdot\mathbf{n}-\rho_0\mathbf{V}_0\times\mathbf{n}\times)\nabla_0 G\,\mathrm{d}B_0, \tag{7}$$

where B is the boundary of domain Ω, $\mathbf{r}(x,y,z)$ is a position vector, the subscript '0' indicates a variable of a differentiation or an integration in the $\mathbf{r}_0(\xi,\eta,\zeta)$ space, $T=1$ if $\mathbf{r}$ is an interior point in Ω and $T=1/2$ if $\mathbf{r}$ is on a smooth boundary, G is the fundamental solution of Laplace's equation. $G=(1/2\pi)\ln R$ for two-dimensional flows and $G=1/4\pi R$ for three-dimensional flows, R is the distance between $\mathbf{r}$ and $\mathbf{r}_0$. In order to establish the complete boundary integral representation for equation (7), we must think of a way to eliminate the domain integral in equation (7). We note that the domain integral in equation (7) can be rewritten as follows by using vector analysis

$$\begin{aligned}\int_\Omega \nabla_0 G\times(\nabla_0\times\rho_0\mathbf{V}_0)\,\mathrm{d}\Omega_0&=\int_\Omega[\nabla_0\times(G\nabla_0\times\rho_0\mathbf{V}_0)-G\nabla_0\times(\nabla_0\times\rho_0\mathbf{V}_0)]\,\mathrm{d}\Omega_0\\&=\int_\Omega\{\nabla_0\times(G\nabla_0\times\rho_0\mathbf{V}_0)-G[\nabla_0(\nabla_0\cdot\rho_0\mathbf{V}_0)-\nabla_0^2(\rho_0\mathbf{V}_0)]\}\,\mathrm{d}\Omega_0\\&=\int_\Omega[\nabla_0\times(G\nabla_0\times\rho_0\mathbf{V}_0)+G\nabla_0^2(\rho_0\mathbf{V}_0)]\,\mathrm{d}\Omega_0\\&=\int_\Omega \mathbf{n}_0\times(G\nabla_0\times\rho_0\mathbf{V}_0)\,\mathrm{d}B_0+\int_B G\nabla_0^2(\rho_0\mathbf{V}_0)\,\mathrm{d}\Omega_0.\end{aligned} \tag{8}$$

According to Green's formula, the domain integral of the second term on the right-hand side of equation (8) can be rewritten as

$$\int_\Omega G\nabla_0^2(\rho_0\mathbf{V}_0)\,\mathrm{d}\Omega_0=\int_B\left(G\frac{\partial(\rho_0\mathbf{V}_0)}{\partial n_0}-\rho_0\mathbf{V}_0\frac{\partial G}{\partial n_0}\right)\mathrm{d}B_0+\int_\Omega\rho_0\mathbf{V}_0\nabla_0^2G\,\mathrm{d}\Omega_0. \tag{9}$$

Since G is the fundamental solution of Laplace's equation, $\nabla^2G=\delta(\mathbf{r}-\mathbf{r}_0)$, δ is the Dirac-delta function. Thus, the second term on the right-hand side of equation (9) is equal to $T\rho\mathbf{V}(\mathbf{r})$. Substituting equations (8) and (9) into equation (7), we have (for convenience the subscript '0' is omitted in the subsequent analysis)

$$\int_B\left(G\frac{\partial\rho\mathbf{V}}{\partial n}-\rho\mathbf{V}\frac{\partial G}{\partial n}\right)\mathrm{d}B+\int_B\mathbf{n}\times(G\nabla\times\rho\mathbf{V})\,\mathrm{d}B+\int_B[(\rho\mathbf{V}\cdot\mathbf{n})-\rho\mathbf{V}\times\mathbf{n}\times]\nabla G\,\mathrm{d}B=0. \tag{10}$$

Once we have known either the boundary value of $\rho\mathbf{V}$ or $\partial\rho\mathbf{V}/\partial n$ of the steady compressible inviscid flow, we may obtain the other one from equation (10). We note that the domain integral on the right-hand side of equation (8) is an integration of the Laplacian operator, with weighting function G. It can be easily transformed into a series of boundary integrals by means of existing methods.[3-4] In order to do this, we first introduce two new functions $\mathbf{A}_0=\nabla^2(\rho\mathbf{V})$ and $\nabla^2G_1=G$. Thus, the domain integral on the right-hand side of equation (8) can be rewritten as

$$\int_\Omega G\nabla^2(\rho\mathbf{V})\,\mathrm{d}\Omega=\int_\Omega\mathbf{A}_0\nabla^2G_1\,\mathrm{d}\Omega=\int_B\left(\mathbf{A}_0\frac{\partial G_1}{\partial n}-G_1\frac{\partial\mathbf{A}_0}{\partial n}\right)\mathrm{d}B+\int_\Omega G_1(\nabla^2\mathbf{A}_0)\mathrm{d}\Omega. \tag{11}$$

Similarly, the domain integral on the right-hand side of equation (11) can also be rewritten as

$$\int_\Omega G_1\nabla^2\mathbf{A}_0\,\mathrm{d}\Omega=\int_B\left(\mathbf{A}_1\frac{\partial G_2}{\partial n}-G_2\frac{\partial\mathbf{A}_1}{\partial n}\right)\mathrm{d}B+\int_\Omega G_2(\nabla^2\mathbf{A}_1)\,\mathrm{d}\Omega,$$

where $\mathbf{A}_1=\nabla^2\mathbf{A}_0$, $\nabla^2G_2=G_1$. The procedure can be generalized by introducing two sequences of functions defined by the following recurrence formulae

$$\mathbf{A}_{j+1}=\nabla^2\mathbf{A}_j,\qquad\nabla^2G_{j+1}=G_j,\quad j=0,1,2,\ldots, \tag{12}$$

where $G_0=G$. Thus, the domain integral on the right-hand side of equation (8) can be expressed as the summation of infinite boundary integrals

$$\int_\Omega G\nabla^2(\rho\mathbf{V})\,\mathrm{d}\Omega=\sum_{j=0}^{\infty}\int_B\left(\mathbf{A}_j\frac{\partial G_{j+1}}{\partial n}-G_{j+1}\frac{\partial\mathbf{A}_j}{\partial n}\right)\mathrm{d}B. \tag{13}$$

Theoretically, on the boundaries, $\mathbf{A}_j$ will finally decrease to zero as j increase to ∞ because $\rho\mathbf{V}$ is finite everywhere on the boundaries and can be approximated by a polynomial.

In order to carry out the special solution of G_{j+1} governed by equation (12), we write the Laplacian operator in terms of a cylindrical (for two-dimensional cases) or spherical (for three-dimensional cases) co-ordinate system. For the special solution G_1, for example, since G is a function of R, the main part of the Laplacian operator that will give the result related to a function of R only will be

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial G_1}{\partial R}\right)\quad\text{for two-dimensional flow,}$$

or

$$\frac{1}{R^2}\frac{\partial}{\partial R}(R^2\partial G_1/\partial R)\quad\text{for three-dimensional flow.}$$

Hence, the special solution G_1 can be obtained from the following ordinary differential equation by direct integration:

Two-dimensional problems

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}\left(R\frac{\mathrm{d}G_1}{\mathrm{d}R}\right)=\frac{1}{2\pi}\ln R,$$

$$G_1=\frac{1}{2\pi}\left[\frac{1}{4}(R^2\ln R-R^2)\right].$$

Three-dimensional problems

$$\frac{1}{R^2}\frac{\mathrm{d}}{\mathrm{d}R}\left(R^2\frac{\mathrm{d}G_1}{\mathrm{d}R}\right)=\frac{1}{4\pi R},$$

$$G_1=\frac{1}{4\pi}\left(\frac{1}{2}R\right).$$

Similarly, we can calculate the special solutions of $G_2, G_3, \ldots$. Finally, we may sum up the following recurrence relationships:

Two-dimensional problems

$$G_j=\frac{1}{2\pi}R^{2j}(C_j\ln R-D_j), \tag{14}$$

where

$$C_{j+1}=C_j/4(j+1)^2, \qquad D_{j+1}=[C_j(j+1)^{-1}+D_j]/4(j+1)^2, \qquad C_0=1, \qquad D_0=0;$$

Three-dimensional problems

$$G_j=R^{(2j-1)}/4\pi(2j)! \tag{15}$$

Notice that formulae (14) and (15) introduce factorials $(j+1)^2$ for two-dimensional flows and $(2j)!$ for three-dimensional flows into the denominators of the coefficients and, hence, guarantee the rapid convergence of the right-hand side of equation (13). Substituting equation (13) into equation (8) and then into equation (7), we obtain the complete boundary integral formulation for steady compressible inviscid flows for $\rho\mathbf{V}$ as follows:

$$T\rho\mathbf{V}(\mathbf{r})=\int_B \mathbf{n}\times(G\nabla\times\rho\mathbf{V})\,\mathrm{d}B+\sum_{j=0}^{\infty}\int_B \mathbf{A}_j\partial G_{j+1}/\partial n-G_{j+1}\partial\mathbf{A}_j/\partial n)\,\mathrm{d}B$$
$$+\int_B(\rho\mathbf{V}\cdot\mathbf{n}-\rho\mathbf{V}\times\mathbf{n}\times)\nabla G\,\mathrm{d}B. \tag{16}$$

The first two terms on the right-hand side of equation (16) represent the non-linear effects due to compressibility. Since the integrals in equation (15) are all boundary integrals, they can be evaluated numerically and iteratively by subdividing the boundary B into boundary elements, as usual in the boundary element method.

NUMERICAL EXAMPLES AND CONCLUDING REMARKS

Numerical examples for compressible inviscid flows about aerofoils and wings at different Mach number and different angles of attack have been calculated by using the present method. It is shown that good surface pressure coefficient C_p results are obtained even if j is less than 2. Figures 1–4 present a few of these results and show the general trend of the comparison between the present method and other numerical methods. From equation (12) we have $\Delta^{j+1}G_j = \delta(\mathbf{r}-\mathbf{r}_0)$. Therefore, G_j is the fundamental solution of Laplacian operator equation with order $(j+1)$.

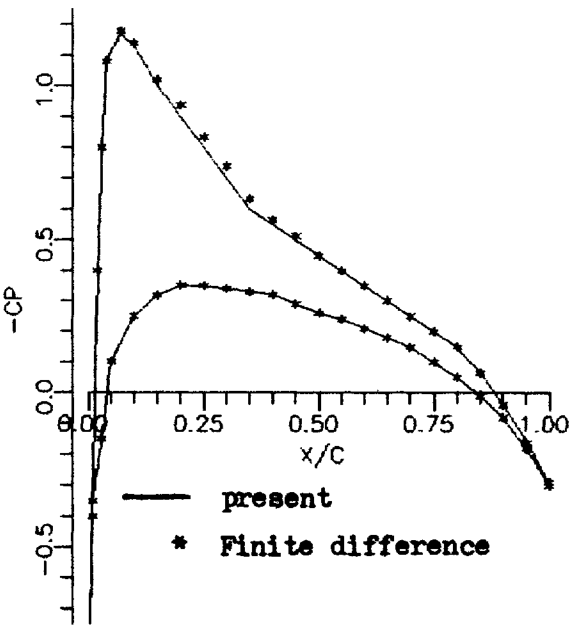

Figure 1. NACA 0012 aerofoil: $Ma_\infty = 0{\cdot}63$, $\alpha = 2°$

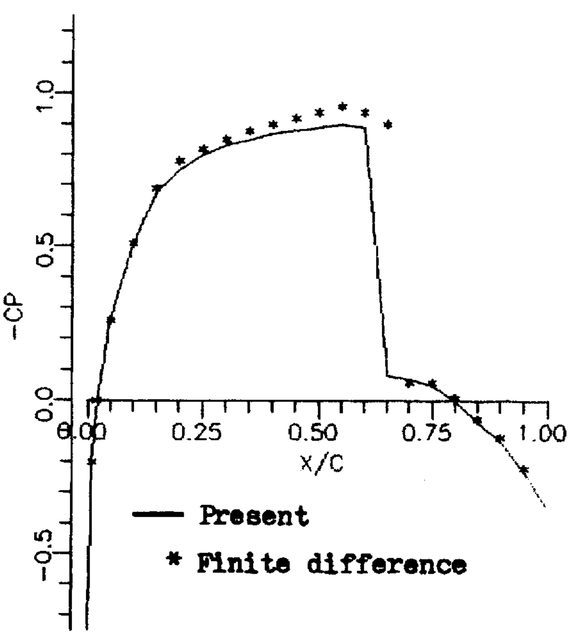

Figure 2. NACA 0012 aerofoil: $Ma_\infty = 0{\cdot}806$, $\alpha = 0°$

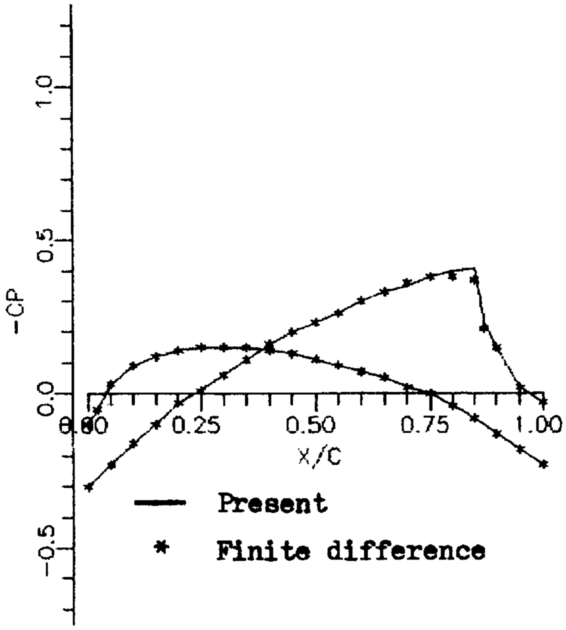

Figure 3. Constant chord 30° swept wing, aspect ratio 4, biconvex aerofoil: $Ma_\infty = 0{\cdot}908$, $\alpha = 0°$

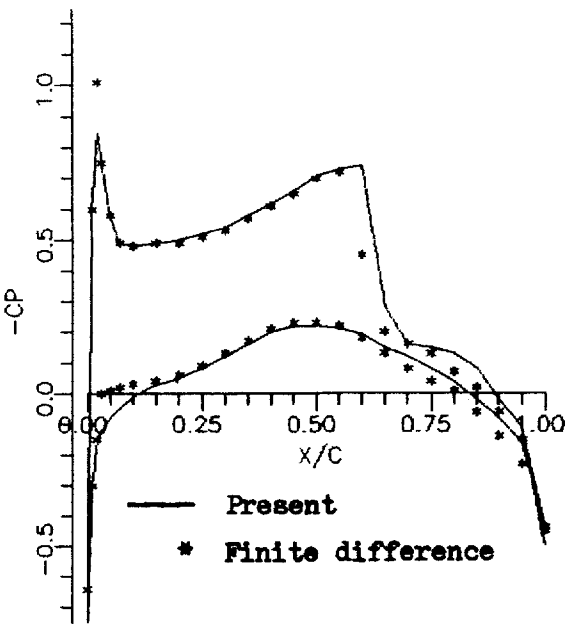

Figure 4. M6 wing: $Ma_\infty = 0{\cdot}84$, $\alpha = 3°$, 20% semi-span

Substituting equation (10) into equation (16), we obtain

$$T\rho\mathbf{V}(\mathbf{r})=\int_{\mathrm{B}}\left(\rho\mathbf{V}\frac{\partial G}{\partial n}-G\frac{\partial(\rho\mathbf{V})}{\partial n}\right)\mathrm{d}\mathbf{B}+\sum_{j=0}^{\infty}\int_{\mathrm{B}}\left(\mathbf{A}_j\frac{\partial\mathbf{A}_{j+1}}{\partial n}-G_{j+1}\frac{\partial\mathbf{A}_j}{\partial n}\right)\mathrm{d}B. \tag{17}$$

The integrand in the first term of the right-hand side of equation (17) is the distribution of first-order fundamental solutions (sources with strength $\partial\rho\mathbf{V}/\partial n$ and doublets with strength $\rho\mathbf{V}$) along boundary B and the integrands in the other terms are the distributions of higher-order fundamental solutions. Hence, equation (17) is really the fundamental-solution distributions for steady compressible inviscid flows governed by non-linear potential equation.

Thus, we have succeeded in establishing the fundamental-solution method for compressible inviscid flows governed by non-linear equations.

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